

Generalising Wigner's theorem

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Abstract

We analyse linear maps of operator algebras $\mathcal{B}_H(\mathcal{H})$ mapping the set of rank- k projectors onto the set of rank- l projectors surjectively. A complete characterisation of such maps for prime $n = \dim \mathcal{H}$ is provided. A particular case corresponding to $k = l = 1$ is well known as the Wigner's theorem. Hence our result may be considered as a generalization of this celebrated Wigner's result.

1 Introduction

The celebrated Wigner's theorem [1] in its original formulation says that any map Φ between rank-1 projectors in a Hilbert space preserving the Hilbert-Schmidt product, i.e. $(\Phi(P_1), \Phi(P_2))_{\text{HS}} = (P_1, P_2)_{\text{HS}}$, is of the form:

$$\Phi(X) = UXU^\dagger \quad \text{or} \quad \Phi(X) = UX^tU^\dagger, \quad (1)$$

where X^t denotes a transposition with respect to a fixed orthonormal basis in \mathcal{H} and U is a unitary operator (see also [2] and recent analysis in [3]). It is clear that any such map induces unitary or antiunitary operation in the original Hilbert space. The Wigner's theorem is sometimes reformulated as follows [6]: one restricts to linear maps $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ which are not necessarily Hilbert-Schmidt isometries. Now, any such map that maps bijectively rank-1 projectors to rank-1 projectors is of the form (1). Clearly, a map mapping rank-1 projectors to rank-1 projectors is by construction a positive map [4, 5] and hence the Wigner's theorem states that a positive trace-preserving map Φ has a positive inverse if and only if it has a form (1).

In this paper we consider linear maps mapping surjectively rank- k projectors to rank- l projectors. We show for which k, l such maps exist and that for prime dimensions of the Hilbert space they have exactly the form (1). We also provide an example of a map which is not of the form (1). Interestingly, linear maps acting surjectively between sets of projectors of fixed rank maps have recently attracted attention in the problem of entanglement detection [7].

Let us consider a real space of self-adjoint operators $\mathcal{B}_H(\mathcal{H})$. We denote the set of rank- k projectors supported on a subspace $V \subset \mathcal{H}$ by $\mathcal{P}_k(V) \subset \mathcal{B}_H$. It is a smooth manifold of dimension $k(n - k)$. Let Φ be a linear endomorphism of $\mathcal{B}(\mathcal{H})$ and let $\Phi(\mathcal{P}_k(\mathcal{H})) = \mathcal{P}_l(\mathcal{H})$. From the fact, that $\mathcal{P}_l(\mathcal{H})$ spans the whole $\mathcal{B}(\mathcal{H})$, one gets that Φ is a surjective endomorphism, hence a bijection, hence also a bijection between sets $\mathcal{P}_k(\mathcal{H})$ and $\mathcal{P}_l(\mathcal{H})$. This implies, that $l = k$ or $l = n - k$. There is a natural linear isomorphism between $\mathcal{P}_k(\mathcal{H})$ and $\mathcal{P}_{n-k}(\mathcal{H})$:

$$R_k(X) = \frac{1}{k} \mathbb{I} \text{Tr} X - X. \quad (2)$$

Hence any bijection $\Phi : \mathcal{P}_k(\mathcal{H}) \rightarrow \mathcal{P}_{n-k}(\mathcal{H})$ can be represented as a composition of R_k and bijective endomorphism of $\mathcal{P}_k(\mathcal{H})$. Therefore, it suffices to characterise all bijective endomorphisms of $\mathcal{P}_k(\mathcal{H})$.

Note, that this problem belongs to so-called linear preserver problem which deal with characterization of linear operators that leave certain properties or certain subsets in its domain invariant. This program was started already by Frobenius [9]. A well known example of linear preservers are rank preserver, nilpotency preserver and spectrum preserver: Φ is rank preserver iff $\Phi(X) = MXN$ or $\Phi(X) = MX^tN$, where M, N are invertible elements from $M_n(\mathbb{C})$. Φ provides a preserver of nilpotency iff $\Phi(X) = MXN$ or $\Phi(X) = MX^tN$, where $M, N \in M_n(\mathbb{C})$ such that $MN = cI_n$ and $c \in \mathbb{C}$. Finally, Φ is a spectrum preserver iff $\Phi(X) = MXM^{-1}$ or $\Phi(X) = MX^tM^{-1}$.

Clearly, Φ defined in (1) is an example of rank preserver and nilpotency preserver, but in our problem the restriction is weaker - we demand preserving the nilpotency only for a one given value of the rank (the set of nilpotent hermitian operators splits into connectivity components grouping the projectors of the same rank). With a weaker assumption one can expect that the resulting set of linear operation can be in general greater. Indeed, for a special choice of n and k we provide an example of an invertible map preserving rank- k projector not being of the form (1).

Recently Marciniak [11] considered a related problem and shown that every positive map Φ such that $\text{rank}\Phi(P) \leq 1$ for any rank-1 projector P is the rank-1 preserver and has a form (1).

2 Main result

This section provides the main result of the paper. Let us start with the following

Proposition 1. *Any $\Phi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ mapping bijectively $\mathcal{P}_k(\mathcal{H})$ onto itself preserves the orthogonality.*

Proof. If $2k > n$, then there are no non-zero mutually orthogonal elements in $\mathcal{P}_k(\mathcal{H})$ and the proposition is true in a trivial way. We will consider the case when $2k \leq n$. Let P_V be the projector onto a $2k$ -dimensional subspace $V \subset \mathcal{H}$. P_V can be decomposed in various ways into the sum of two rank- k orthogonal projectors P_1 and P_2 . They are mapped via Φ onto two rank- k projectors $Q_1 = \Phi(P_1)$ and $Q_2 = \Phi(P_2)$. From the positivity of Q_1 and Q_2 one has $\text{Im}Q_i = \text{Im}\Phi(P_i) \subset \text{Im}\Phi(P_V)$. One can repeat it for any choice of P_1 and P_2 , hence $\forall P \in \mathcal{P}_k(V)$ $\text{Im}\Phi(P) \subset \text{Im}\Phi(P_V)$ and thus $\Phi(\mathcal{P}_k(V)) \subset \mathcal{B}_H(\text{Im}\Phi(P_V))$. Because $\mathcal{P}_k(V)$ spans the whole $\mathcal{B}_H(V)$, we have that $\Phi(\mathcal{B}_H(V)) \subset \mathcal{B}_H(\text{Im}\Phi(P_V))$.

While Φ is bijective on $\mathcal{P}_k(\mathcal{H})$, it is bijective on $\mathcal{B}_H(\mathcal{H})$. Hence $\dim \mathcal{B}_H(V) \leq \dim \mathcal{B}_H(\text{Im}\Phi(P_V))$ and hence $\dim V \leq \dim \text{Im}\Phi(P_V)$. While $\Phi(P_V)$ is a sum of two rank- k projectors, $\dim \text{Im}\Phi(P_V) \leq 2k = \dim V$ and hence $\dim \text{Im}\Phi(P_V) = \dim V$. Φ establishes a bijection between $\mathcal{B}_H(V)$ and $\mathcal{B}_H(\text{Im}\Phi(P_V))$ and hence between $\mathcal{P}_k(V)$ and $\mathcal{P}_k(\text{Im}\Phi(P_V))$. Any rank- k projector $Q \in \mathcal{P}_k(\text{Im}\Phi(P_V))$ can be realised as $\Phi(P)$ for some $P \in \mathcal{P}_k(V)$.

Now we choose the basis of $\text{Im}\Phi(P_V)$ to make $\Phi(P_V)$ diagonal. Take a rank- k dimensional projector $Q \in \mathcal{P}_k(\text{Im}\Phi(P_V))$, diagonal in this base (commuting with $\Phi(P_V)$). The operator $\Phi(P_V) - Q = \Phi(P_V) - \Phi(P) = \Phi(P_V - P)$ is a rank k -projector, diagonal in the chosen basis. One can easily find, that this implies, that only possible values on the diagonal of $\Phi(P_V)$ are 1s and 2s. But the rank of $\Phi(P_V)$ and its trace are equal $2k$, so $\Phi(P_V)$ is the projector onto $\text{Im}\Phi(P_V)$.

For any two orthogonal projectors P_1 and P_2 the operator $\Phi(P_1 + P_2) = \Phi(P_1) + \Phi(P_2)$ is a rank- $2k$ dimensional projector, hence the projectors $\Phi(P_1)$ and $\Phi(P_2)$ are orthogonal \square

One has immediately the following:

Corollary 1. *Any $\Phi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ mapping bijectively $\mathcal{P}_k(\mathcal{H})$ onto itself maps bijectively $\mathcal{P}_{qk}(\mathcal{H})$ onto itself for $q \in \mathbb{N}$.*

Remark 1. Note, that Wigner's theorem immediately follows from preserving the orthogonality relation and the properties of a spectrum preserver. Indeed, preserving the orthogonality relation of rank-1 projectors

implies that a Schatten decomposition $\sum_i \lambda_i P_i$ of a hermitian operator is mapped to another Schatten decomposition with the same spectrum. Such a map is therefore a spectrum preserver and hence has a form $\Phi(X) = MXM^{-1}$ or $\Phi(X) = MX^t M^{-1}$. Finally, preservation of orthogonality implies that M is unitary.

Proposition 2. *Assume, that $k = n \bmod l$. If any linear map $\Psi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ transforming $\mathcal{P}_k(\mathcal{H})$ onto itself bijectively is of the form (1), then also each map $\Phi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ transforming $\mathcal{P}_l(\mathcal{H})$ onto itself bijectively is of the form (1).*

Proof. Let $\Phi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ maps bijectively $\mathcal{P}_l(\mathcal{H})$ onto itself. Let $n = k + q \cdot l$. Then due to Corollary 1, for any $P \in \mathcal{P}_{ql}(\mathcal{H})$ one has $\Phi(P) \in \mathcal{P}_{ql}(\mathcal{H})$. Now, any projector from $\mathcal{P}_{ql}(\mathcal{H})$ may be written as $I - P_k$ with $P_k \in \mathcal{P}_k(\mathcal{H})$ and hence

$$\Phi(I - P_k) = \Phi(I) - \Phi(P_k) =: I - Q_k, \quad (3)$$

for some $Q_k \in \mathcal{P}_k(\mathcal{H})$. One has therefore

$$Q_k = \Phi(P_k) - \Phi(I) + I. \quad (4)$$

The above relation defines a map $P_k \rightarrow Q_k$ transforming $\mathcal{P}_k(\mathcal{H})$ onto itself bijectively, so due to our assumption it can be written as

$$Q_k = U \tilde{P}_k U^\dagger. \quad (5)$$

where $\tilde{X} = X$ or $\tilde{X} = X^t$. Finally

$$\Phi(P_k) = \Phi(I) - I + U \tilde{P}_k U^\dagger, \quad (6)$$

and by linearity it may be extended to the following linear map on $\mathcal{B}(\mathcal{H})$

$$\Phi(X) = \frac{1}{k} [\Phi(I) - I] \text{Tr} X + U \tilde{X} U^\dagger. \quad (7)$$

In particular if $X = P_l \in \mathcal{P}_l(\mathcal{H})$ one has

$$\Phi(P_l) = \frac{l}{k} [\Phi(I) - I] + Q_l =: D + Q_l, \quad (8)$$

where $Q_l = U \tilde{P}_l U^\dagger \in \mathcal{P}_l(\mathcal{H})$. To complete the proof we need to show that $D = 0$. Since $D + Q_l$ is a projector one has $(D + Q_l)^2 = D + Q_l$ and hence

$$DQ_l + Q_l D = D - D^2. \quad (9)$$

Taking into account that Φ is trace-preserving one has $\text{Tr} D = 0$ which implies $2\text{Tr}(DQ_l) = -\text{Tr} D^2$ for all $Q_l \in \mathcal{P}_l(\mathcal{H})$. Now, if Q_l is a projector on the l -dim. subspace spanned by eigenvectors of D corresponding to l largest eigenvalues of D , i.e $d_1 \geq d_2 \geq \dots \geq d_l$, then $2\text{Tr}(DQ_l) = 2 \sum_{i=1}^l d_i = -\text{Tr} D^2 \leq 0$ but since D is traceless one has $\sum_{i=1}^l d_i \geq 0$ which proves that $D = 0$. \square

The main result of the paper is provided by the following

Theorem 1. *If n is prime, then any Φ mapping surjectively rank- k projectors into rank- k projectors is of the form (1).*

Proof. Let us define a sequence via formula $k_{i+1} = n \bmod k_i$ and $k_0 = k$. This is strictly decreasing, finite sequence, and it terminates at 0 and let $0 = n \bmod k_*$, that is, the sequence reads $\{k_0 = k > k_1 > k_2 > \dots > k_* > 0\}$. Due to the Proposition 2, if any $\Psi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ maps bijectively $\mathcal{P}_{k_{i+1}}(\mathcal{H})$ onto itself is of the form (1), then any $\Psi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ mapping bijectively $\mathcal{P}_{k_i}(\mathcal{H})$ onto itself is of the form (1) as well. Now, due to the Wigner's theorem, if $k_* = 1$ then for any $k_i > k_*$ in the sequence, any $\Psi : \mathcal{B}_H(\mathcal{H}) \rightarrow \mathcal{B}_H(\mathcal{H})$ mapping bijectively $\mathcal{P}_{k_i}(\mathcal{H})$ onto itself is of the form (1), in particular $k_0 = k$. Note, that k_* is by construction a divisor of n and hence if n is prime, then $k_* = 1$ which complete the proof. \square

Observe, that $n = q_i k_i + k_{i+1}$ and if for some number d one has $d|n$ and $d|k_i$ then $d|k_{i+1}$, so the common divisors of n and $k_0 = k$ are also the common divisors of all elements of the sequence. Thus this method does not give a conclusive answer if the starting point $k_0 = k$ is not relatively prime to n . If n and k are relatively prime then a conclusive answer is not guaranteed. Indeed, if $k = 3$ and $n = 10$ one has the conclusive answer, but for $k = 3$ and $n = 8$ the method does not give a conclusive answer. Having in mind, that k_* is by construction a divisor of n , one has the following

Remark 2. To complete the characterisation of surjective maps from $\mathcal{P}_k(\mathcal{H})$ into $\mathcal{P}_k(\mathcal{H})$ it is enough to characterise these maps for $k|n$.

Remark 3. Let $n = 2k$. Consider

$$\Phi(X) = \frac{\mathbb{I}_{2k}}{k} \text{Tr} X - X. \quad (10)$$

It is clear that Φ maps rank- k projectors into rank- k projectors but evidently it does not have the form (1). Indeed, if P is rank-1 projector then $\Phi(P)$ has rank $n - 1$ and hence it is not rank-1 projector. Note, that $\Phi^{-1} = \Phi$.

Such maps are the already defined (2) and because $\mathcal{P}_k(\mathcal{H})$ and $\mathcal{P}_{n-k}(\mathcal{H})$ are the same set, these maps are endomorphisms. This encourages us to claim that these are the only additional endomorphisms. Therefore we pose the following

Conjecture 1. Let $n = 2k$ and $\Phi : \mathcal{P}_k(\mathcal{H}) \rightarrow \mathcal{P}_k(\mathcal{H})$ surjectively. Then the map is of the form Φ or $\Phi \circ R_k$, where Φ is of the form (1) and R_k is defined by (2).

This conjecture allows to perform perfect characterization of surjective maps $\mathcal{P}_k(\mathcal{H}) \rightarrow \mathcal{P}_k(\mathcal{H})$

Proposition 3. Let us assume that Conjecture 1 is true. If $k|n$, $n/k > 2$ and $\Phi : \mathcal{P}_k(\mathcal{H}) \rightarrow \mathcal{P}_k(\mathcal{H})$ is surjective, then Φ has the form (1).

Proof. Let $\{e_i\}_{i=0}^{n-1}$ be the standard basis of \mathcal{H} and let P_i be a rank- k projector defined by

$$P_i = \sum_{j=0}^{k-1} |e_{ik+j}\rangle \langle e_{ik+j}|, \quad i = 0, 1, \dots, \frac{n}{k} - 1.$$

It is clear that P_i and P_j are mutually orthogonal for $i \neq j$. Now, due to the Proposition 1, a set of projectors $\{P_i\}$ is mapped to the set of pairwise orthogonal rank- k projectors. Now, since

$$P_0 + \dots + P_{n/k} = \mathbb{I} = \Phi(P_0) + \dots + \Phi(P_{n/k})$$

projectors $\{\Phi(P_0), \dots, \Phi(P_{n/k})\}$ are unitarily equivalent to $\{P_0, \dots, P_{n/k}\}$, that is, $\Phi(P_i) = U P_i U^\dagger$ for some unitary U . Without loosing generality we may assume that $U = \mathbb{I}$, that is, Φ maps P_i to P_i . Let $V_i = \text{span}\{e_{k \cdot i}, \dots, e_{k \cdot i + (k-1)}\}$ be the range of P_i and let \mathbb{I}_{ij} be the projector onto the subspace $V_i \oplus V_j$. Let

$$\Phi_{ij} := \mathbb{I}_{ij} \Phi \mathbb{I}_{ij},$$

be a restriction of Φ to the subspace $V_i \oplus V_j$. By the Conjecture 1 it is of the form $\Phi_{ij}(X) = U_{ij} \tilde{\Phi}_{ij}(X) U_{ij}^\dagger$, where $\tilde{\Phi}_{ij}$ maps X to $X, X^t, \frac{1}{k} \mathbb{I}_{ij} \text{Tr} X - X$ or $\frac{1}{k} \mathbb{I}_{ij} \text{Tr} X - X^t$. One has that for all i, j $\Phi_{ij}(P_i) = P_i$ and $\Phi_{ij}(P_j) = P_j$, hence for all α, β

$$\begin{bmatrix} \alpha P_i & \\ & \beta P_j \end{bmatrix} = U_{ij} \begin{bmatrix} \alpha P_i & \\ & \beta P_j \end{bmatrix} U_{ij}^\dagger = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha P_i & \\ & \beta P_j \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger$$

what implies that U_{ij} is block-diagonal if $\tilde{\Phi}_{ij}(X) = X$ or X^t and U_{ij} is block-antidiagonal if $\tilde{\Phi}_{ij}(X) = \frac{1}{k} \mathbb{I}_{ij} \text{Tr} X - X$ or $\frac{1}{k} \mathbb{I}_{ij} \text{Tr} X - X^t$.

Moreover, if $\psi \in V_i$ then

$$\Phi_{ij}(|\psi\rangle\langle\psi|) = \mathbb{I}_{ij}\Phi(|\psi\rangle\langle\psi|)$$

does not depend on j , and similarly if $\phi \in V_j$ then

$$\Phi_{ij}(|\phi\rangle\langle\phi|) = \mathbb{I}_{ij}\Phi(|\phi\rangle\langle\phi|)$$

does not depend on i . It follows, that all Φ_{ij} does nor depend on i, j and hence $\tilde{\Phi}_{ij}(X) = X$ or X^t (otherwise it could not give the same result for different j s if $n/k > 2$, as the reduction map has the information about the trace of the second block). Now we know that all U_{ij} s are block-diagonal and again, because $\Phi_{ij}(|\Psi\rangle\langle\Psi|)$ has to give the same result for all j s one gets that $U_{ij} = U_i \oplus U_j$.

Finally we get that

$$\Phi(X) = U X U^\dagger \quad \text{or} \quad \Phi(X) = U X^t U^\dagger$$

with $U = \bigoplus_{i=1}^{n/k} U_i$, which ends the proof. \square

3 Conclusions

Let us summarise the paper by the following remarks:

Remark 4. Let us observe that if we relax the condition that the map Φ is invertible then one may have maps from $\mathcal{P}_k(\mathbb{C}^n)$ to $\mathcal{P}_l(\mathbb{C}^n)$ with $l \neq n - k$. The well known example is provided by the Breuer-Hall map $\Phi_{\text{BH}} : M_{2n}(\mathbb{C}) \rightarrow M_{2n}(\mathbb{C})$ defined as follows [12, 13, 14]

$$\Phi_{\text{BH}}(X) = \frac{1}{2(n-1)} (\mathbb{I}_{2n} \text{Tr} X - X - U X^t U^\dagger), \quad (11)$$

where U is an arbitrary anti-symmetric $2n \times 2n$ matrix. Φ_{BH} maps rank-1 projectors into projectors of rank $2(n-1)$. It is evident that Φ_{BH} is not invertible.

Remark 5. Let us observe that maps (1) are characterized by the following property: Φ is positive and trace-preserving and $\Phi^{-1} = \Phi^*$, where the dual map Φ^* is defined by $(X, \Phi(Y))_{\text{HS}} = (\Phi^*(X), Y)_{\text{HS}}$. Interestingly, the map (10) is characterized by the following property: Φ is trace-preserving and $\Phi^{-1} = \Phi^* = \Phi$. It means that both (1) and (10) are isometries with respect Hilbert-Schmidt product and hence the corresponding eigenvalues satisfy $|\lambda_i| = 1$.

On the other hand, any rank-1 projector P can be decomposed as a combination of rank- k projectors commuting with P and hence any hermitian operator can be decomposed as a combination of n commuting rank- k projectors. This combination is mapped into another combination of rank- k projectors (in general not commuting) with the same coefficients and hence there exists a common upper bound for the maximal eigenvalue of all invertible maps mapping $\mathcal{P}_k(\mathcal{H})$ to itself. While such maps form a group, their eigenvalues has to lie therefore on the unit circle and at least one of them is equal 1. We stress that this property is not equivalent to being a Hilbert-Schmidt isometry. If one could prove that for any invertible map Φ mapping $\mathcal{P}_k(\mathcal{H})$ to itself its adjoint map Φ^* has the same property, then it would imply that this class is a subclass of Hilbert-Schmidt isometries.

Remark 6. The crucial difference between (1) and (10) is positivity. Let us recall [5] that a trace-preserving map is positive iff

$$\|\Phi(X)\|_1 \leq \|X\|_1, \quad (12)$$

for all self-adjoint elements X . It is clear that (10) is not positive and hence violates (12). Interestingly, for X traceless one has

$$\|\Phi(X)\|_1 = \|-X\|_1 = \|X\|_1. \quad (13)$$

In particular taking two density operators ρ_1 and ρ_2 one has

$$||\Phi(\rho_1 - \rho_2)||_1 = ||\rho_1 - \rho_2||_1 , \quad (14)$$

which means that (1) preserves the distinguishability between arbitrary quantum states. Clearly, (1) enjoys the same property.

Finally, it is hoped that presented result finds applications in quantum information theory (e.g. entanglement detection) and the analysis of symmetries of quantum systems (e.g the evolution of quantum systems).

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